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ORIGINAL PAPER

Li–Yorke chaos in a coupled lattice system related with Belusov–Zhabotinskii reaction

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Abstract Garca Guirao and Lampart (J Math Chem 48:66–71, 2010; J Math Chem 48:159–164, 2010) said that for non-zero couplings constant, the lattice dynamical system is more complicated. Motivated by this, in this paper, we prove that this coupled map lattice system is Li–Yorke chaotic for coupling constant $0 < \epsilon < 1$.

Keywords Coupled map lattice · Li–Yorke chaos · Topological chaos · Horseshoe

1 Introduction

By a dynamical system, we mean a pair (X, f), where X is a compact metric space and $f : X \longrightarrow X$ is continuous. Since the introduction of the term of chaos in 1975 by Li and Yorke [13], known as Li–Yorke chaos today, dynamical properties were highly consider in the literature (see e.g., [2,6]) because are good examples of problems coming from the theory of Topological Dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

In many physical/chemical engineering applications such as digital filtering, imaging and spatial dynamical system, dynamical systems have recently appeared as an important subject for investigation (see e.g., [5,12,15,16]). In [9], Guirao and Lampart studied a lattice dynamical system and proved it is chaotic in the sense of Li–Yorke, Devaney and positive entropy, motivated by their research, we shall further investigate into the dynamical properties of general lattice dynamical system.

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2 Preliminaries

First, Let us recall the notions of Li–Yorke chaos and positive topological entropy which is known to topological chaos. Throughout this paper, I denotes the unite closed interval [0, 1].

Definition 1 A pair of points $x, y \in X$ is called a Li–Yorke pair of system (X, f) if

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

A subset $\mathcal{S} \subset X$ is called a scrambled set of f if $\sharp\mathcal{S} \geq 2$ and every pair of distinct points in $\mathcal{S}$ is a Li–Yorke pair. According to Li and Yorke [13], (X, f) is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set.

In 1986, Smítal [14] proved an important result for Li–Yorke chaos:

Proposition 1 *Dynamical system (I, f) is Li–Yorke chaotic if and only if it has a Li–Yorke pair.*

An attempt to measure the complexity of a dynamical system is based on a computation of how many points are necessary in order to approximate (in some sense) with their orbits all possible orbits of the system. A formalization of this intuition leads to the notion of topological entropy of the map f, which is due to Adler, Konheim and McAndrew [1]. We recall here the equivalent definition formulated by Bowen [7], and independently by Dinaburg [3]: the topological entropy of a map f is a number $h(f) \in [0, +\infty]$ defined by

$$h(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \sharp E(n, f, \epsilon)$$

where $E(n, f, \epsilon)$ is a (n, f, ϵ)-span with minimal possible number of points.

A map f is topologically chaotic if its topological entropy $h(f)$ is positive. Recently, the following result was proved by Blanchard et al. [4].

Lemma 1 *If the topological entropy of a system (X, f) is positive then there exists a scrambled Cantor set, in particular f is chaotic in the sense of Li–Yorke.*

Recall that the interval map f has a horseshoe if there exist two closed non-degenerate subintervals J, K with disjoint interiors such that $f(J) \cap f(K) \supset J \cup K$. The next theorem sums up two results on horseshoe. The first point is due to Block and Coppel, and the second one derives from [2].

Theorem 1 *Let f be an interval map with a horseshoe. Then*

(1-1) f *is of type 3 for sharkovskii's order.*
(1-2) $h(f) \geq \log 2$.

3 Applications

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \left\{ x : x = \{x_i\}, \quad x_i \in \mathbb{R}^d, \quad i \in \mathbb{Z}^D, \quad \|x_i\| < \infty \right\},$$

where $d \geq 1$ is the dimension of the range space of the map of state x_i, $D \geq 1$ is the dimension of the lattice and the l^2 norm $\|x\|_2 = \left(\sum_{i \in \mathbb{Z}^D} : |x_i|^2\right)^{1/2}$ is usually taken ($|x_i|$ is the length of the vector x_i).

Now we deal with a system coming from a lattice dynamical system stated by Kaneko in [10,11] which is related to the Belusov–Zhabotinskii reaction:

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2 \left[f\left(x_{n-1}^m\right) + f\left(x_{n+1}^m\right) \right] \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), ϵ is coupling constant and $f(x)$ is the unimodal map on the unite closed interval $I = [0, 1]$, i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$.

In general, one of the following periodic boundary conditions of the system (1) is assumed:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, \ldots, x_L), (y_1, \ldots, y_L)) = \sqrt{\sum_{i=1}^{L} |x_i - y_i|^2}$$

for any $(x_1, \ldots, x_L), (y_1, \ldots, y_L) \in I^L$.

Define the map $F : (I^L, d) \longrightarrow (I^L, d)$ by $F(x_1, \ldots, x_L) = (y_1, \ldots, y_L)$ where $y_i = (1-\epsilon) f(x_i) + \epsilon/2[f(x_{i-1}) + f(x_{i+1}^m)]$. Clearly, system (1) is equivalent to system (I^L, F).

The Eq. (1) was studied by many authors, mostly experimentally or semi-analytically than analytically. In [8,9], the authors proved that if f is the tent map, defined by

$$f(x) = \begin{cases} 2x, & 0 \leq i < 1/2, \\ 2 - 2x, & 1/2 \leq i \leq 1, \end{cases}$$

then coupled map lattice system is chaotic in the sense of Li–Yorke, Devaney and positive topological entropy for zero coupling constant. Now we shall further investigate into this system.

Theorem 2 *Let f be the unimodal map on the unite closed interval I. Then f is chaotic in the sense of Li–Yorke and $h(f) \geq \log 2$.*

Proof As f is a unimodal map, then we can set $J = [0, c]$ and $K = [c, 1]$, where c is the unique critical point of f such that $f(c) = 1$. It is easy to see that $f(J) \cap f(K) = I \supset J \cup K$. So f has a horseshoe. Combining this with Lemma 1 and Theorem 1, it follows that f is chaotic in the sense of Li–Yorke and $h(f) \geq \log 2$. □

Theorem 3 *Let f be the unimodal map on I. Then the system*

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2 \left[f\left(x_{n-1}^m\right) + f\left(x_{n+1}^m\right)\right] \tag{2}$$

is Li–Yorke chaotic for any $0 < \epsilon < 1$.

Proof According to theorem 2, we know that f has an uncountable scrambled set $\mathcal{S} \subset I$. Set $\mathcal{D} = \left\{(x_1, x_2, \ldots, x_L) \in I^L : x_1 = x_2 = \cdots = x_L \in \mathcal{S}\right\}$.

Now we assert that $\mathcal{D}$ is a scrambled set of system (I^L, F).

In fact, for any pair of distinct points $\overrightarrow{x} = (x, \ldots, x)$, $\overrightarrow{y} = (y, \ldots, y) \in \mathcal{D}$ and any positive integer n, it is easy to see that

$$F^n(\overrightarrow{x}) = (f^n(x), \ldots, f^n(x))$$

and

$$F^n(\overrightarrow{y}) = (f^n(y), \ldots, f^n(y))$$

Noting the fact that (x, y) is a Li–Yorke pair of f, it is not difficult to check that

$$\limsup_{n\to\infty} d(F^n(\overrightarrow{x}), F^n(\overrightarrow{y})) = \sqrt{L} \cdot \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$$

and

$$\liminf_{n\to\infty} d(F^n(\overrightarrow{x}), F^n(\overrightarrow{y})) = \sqrt{L} \cdot \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$$

This implies that $(\overrightarrow{x}, \overrightarrow{y})$ is a Li–Yorke pair of F.

Hence, (I^L, F) (or, system (2)) is Li–Yorke chaotic as $\mathcal{D}$ is uncountable. □

4 Concluding remarks

Applying Theorem 2 and the equation $h\left(\prod_{i=1}^{L} f_i\right) = \sum_{i=1}^{L} h(f_i)$, we have that for each unimodal map f, the topological entropy of system (1) is not less than $L \log 2$ when $\epsilon = 0$.

In [9], the authors proved that system (1) is Devaney chaotic for tent map and $\epsilon = 0$. Here, we shall show that this does not hold for general unimodal maps. For example, let us put $f : I \longrightarrow I$ such that

$$f(x) = \begin{cases} 0, & 0 \le i < 1/4, \\ 4x - 1, & 1/4 \le i < 1/2, \\ 2 - 2x, & 1/2 \le i \le 1, \end{cases}$$

Clearly, f is a unimodal map. Take $\mathcal{U} = [0, 1/4)^L$ and $\mathcal{V} = (1/2, 1]^L$, then both $\mathcal{U}$ and $\mathcal{V}$ are open subsets of I^L. And for each positive integer n, $F^n(\mathcal{U}) \cap \mathcal{V} = \{(0, \dots, 0)\} \cap \mathcal{V} = \emptyset$. So system (1) is not Devaney chaotic.

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